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Deformation of algebras and related quantizations*

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Abstract. Weak Bose- and Fermi-type quantizations are defined via deformations of algebras generated by characters from linear topological spaces into commutative or Grassmann algebras, respectively. This yields an inherent symmetrization and antisymmetrization, respectively, of deformed products with respect to undeformed products. Combining both quantizations one obtains a weak supersymmetric quantization of a $\mathbb{Z}_2$-graded algebra. For Bose-type quantizations a Hilbert space representation based on quasi-invariant measures is outlined.

1. Introduction

Deformations as a tool for quantization schemes have been in use for about 20 years (see [1–11]). However, these publications are exclusively concerned with quantizations which, in our terminology, are those of Bose type. This is most probably due to the lack of a widely accepted convincing description of non-quantized classical dynamical systems (including relativistic spinor equations) in the framework of Grassmann algebras (see [11–13]). An additional reason might be the need for more sophisticated techniques for integration in (topological) Grassmann algebras. Since for our purpose it suffices to use Grassmann algebras in a rather casual way, vaguely based on Bryce DeWitt's introduction of infinite-dimensional Grassmann algebras (see [13] for an excellent introduction to this and related topics), we have abstained here from going into detail on this topic. In this paper we present two approaches to quantization by deformation. One that is based on of what we have called d-contractions (these are closely related to inner products of Clifford algebras, see [11]), the other one uses a generalized concept of characters and projective representations of groups generated by these characters. These schemes are equivalent, their difference being purely technical. That is, the choice between them depends on which particular problem one wants to consider. Both concepts have in common that they allow us to basically treat Bose- and Fermi-type quantizations in the same setting. This provides a general basis for a supersymmetric quantization with either of these schemes. In an appendix we have outlined a representation of Bose-type quantizations with infinitely many degrees of freedom on Hilbert spaces with quasi-invariant measures, following thereby closely a construction first provided by Gel'fand and co-workers (see [17]). Notwithstanding a few sketchy examples which refer to physics the paper is mainly concerned with mathematical statements. For this reasons it is organized in a predominantly mathematical fashion (lemma, corollary, theorem, etc). This is not meant to be *l'art pour l'art* but rather to outline the mathematical framework in an appropriate way.

* Dedicated to G Pickert on the occasion of his 80th birthday.

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For lack of a complete collection of literature dealing with the subjects treated here, it may happen that unintentionally some relevant contributions of other authors have not been taken into regard. If this is the case no claim to priority or originality is made.

2. Deformation by contraction

Definition 1. *Let $\mathbb{A}$ be an algebra over $\mathbb{C}$ with an involution $AB \to B^*A^*$ which is generated by a unit element $\mathbf{1}_{\mathbb{A}}$ and a set of elements $a(j)$, $j \in J$. Let $\mathbb{A}^{(m)}$, $m \in \mathbb{N}$, be the complex linear hull of monomials $a(j_1)\cdots a(j_m)$, $(j_1, \ldots, j_m) \in J^m$ and let $\mathbb{A}^{(0)} = \mathbb{C}\mathbf{1}_{\mathbb{A}}$. Let $A_1, A_2, A_3 \in \mathbb{A}$ be arbitrary monomials, and let $K_d : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ be a bilinear map which is defined as follows:*

$$K_d : \mathbb{A}^{(m)} \times \mathbb{A}^{(n)} \to \bigoplus_{j=1}^{[(m+n)/2]} \mathbb{A}^{(m+n-2j)} \qquad \textit{for} \quad mn \neq 0 \tag{1}$$

$$K_d : \mathbb{A}^{(m)} \times \mathbb{A}^{(n)} \to \{0\} \qquad \textit{for} \quad mn = 0 \tag{2}$$

$$\begin{aligned} &K_d(A_1, A_2A_3) + A_1K_d(A_2, A_3) + K_d(A_1, K_d(A_2, A_3)) \\ &\qquad = K_d(A_1A_2, A_3) + K_d(A_1, A_2)A_3 + K_d(K_d(A_1, A_2), A_3) \end{aligned} \tag{3}$$

$$K_d(A_1, A_2) = K_d(A_2^*, A_1^*)^*. \tag{4}$$

By defining a binary operation,

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A} : (A, B) \to A \circ B = AB + K_d(A, B) \tag{5}$$

$\mathbb{A}$ (as a linear space) becomes by a standard result in the Gerstenhaber deformation theory of algebras (cf [1]) an associative algebra over $\mathbb{C}$ with a unit element $\mathbf{1}_{\mathbb{A}}$ and an involution $A \circ B \to B^* \circ A^*$.

The map defined by (5) will be called a deformation of the algebra $\mathbb{A}$ induced by the map K_d; the latter will be called a d-contraction for $\mathbb{A}$.

3. Deformation by a projective representation

We shall now present another approach to deformations namely quantizations which use a projective representation of groups of so-called $\mathbb{A}$-valued characters. Both this approach and that defined above are equivalent. The difference is rather technical, depending on which particular problem one wants to consider.

Definition 2. *Let $\mathbb{A}$ be an algebra over $\mathbb{C}$ with a unit element $\mathbf{1}_{\mathbb{A}}$, and let X be a linear topological space. A map $W : X \to \mathbb{A} : f \to W(f)$ shall be called an $\mathbb{A}$-valued character of X (in short, an $\mathbb{A}$-character) if there holds:*

(a) $W(\alpha f)W(\beta f) = W((\alpha + \beta)f)\, \forall f \in X, \forall\alpha, \beta \in \mathbb{C}$;
(b) $W(0) = \mathbf{1}_{\mathbb{A}}$;
(c) $w : X \to \mathbb{A} : f \to w(f) := \lim_{\alpha\to 0}[(-W(0) + W(\alpha f))/\alpha] \equiv \partial_\alpha W(\alpha f)|_{\alpha=0}, \alpha \in \mathbb{C}$, *is a linear map (the limit being defined in some topology).*

Remark 1. The reason for calling W a character is obvious. If, for example, $\mathbb{A}$ is an algebra of complex-valued functions $f : \mathbb{R}^N \to \mathbb{C}$ and $W(x) = \exp[\mathrm{i}xw]$ $x \in \mathbb{R}^N$, $w \in \mathbb{R}^N$, then W is a character in the usual sense. The generalization given here will become useful when considering Grassmann algebras. Note that $W(f)W(g)$ is not required to be equal to $W(f+g)$ unless f and g are linearly dependent.

Corollary 1. *The set of* $W(z, f) \equiv \mathrm{e}^z W(f), z \in \mathbb{C}, f \in X$, *constitutes with respect to the multiplication in* $\mathbb{A}$ *a group which shall be denoted by* $\mathcal{G}_W(\mathbb{A}, X)$.

Lemma 1. *Let* $\mathbb{A}, X$ *and* W *be as in definition 2, and let* $G : X \times X \to \mathbb{C}$ *be a bilinear map. Let* $f_1, f_2, \ldots, f_{m+n}$ *be any finite sequence of elements in* X, *let* $W1 \equiv W(z_1, f_1) \cdots W(z_m, f_m)$, $W2 \equiv W(z_{m+1}, f_{m+1}) \cdots W(z_{m+n}, f_{m+n})$, *and let* $\overline{h}$ *be a real or complex parameter. Then the binary relation defined by*

$$W1 \circ W2 := \exp\left[\overline{h} \sum_{j=1}^{m} \sum_{k=1}^{n} G(f_j, f_{m+k})\right] W1W2$$

is associative.

Proof. Let $W3 \equiv W(z_{m+n+1}, f_{m+n+1}) \cdots W(z_{m+n+p}, f_{m+n+p})$, and further let

$$\sigma(1,2) \equiv \overline{h} \sum_{j=1}^{m} \sum_{k=1}^{n} G(f_j, f_{m+k})$$

$$\sigma(1,3) \equiv \overline{h} \sum_{j=1}^{m} \sum_{l=1}^{p} G(f_j, f_{m+n+l})$$

$$\sigma(2,3) \equiv \overline{h} \sum_{k=1}^{n} \sum_{l=1}^{p} G(f_{m+k}, f_{m+n+l})$$

$$\sigma(12,3) \equiv \overline{h} \sum_{j=1}^{m+n} \sum_{l=1}^{p} G(f_j, f_{m+n+l})$$

$$\sigma(1,23) \equiv \overline{h} \sum_{j=1}^{m} \sum_{k=1}^{n+p} G(f_j, f_{m+k}).$$

Then $\sigma(12,3) = \sigma(1,3) + \sigma(2,3)$ and $\sigma(1,23) = \sigma(1,2) + \sigma(1,3)$. Hence $(W1 \circ W2) \circ W3 = \exp[\sigma(1,2) + \sigma(12,3)] W_1 W_2 W_3 = \exp[\sigma(1,2) + \sigma(1,3) + \sigma(2,3)] W_1 W_2 W_3 = \exp[\sigma(2,3) + \sigma(1,23)] W_1 W_2 W_3 = W1 \circ (W2 \circ W3)$. $\square$

Corollary 2. *The set of* $W(z, f), z \in \mathbb{C}, f \in X$, *together with the multiplication introduced in lemma 1 is a group which shall be denoted by* $\mathcal{G}_W(\mathbb{A}, X, \overline{h}G)$.

Remark 2. Let $F(j, m) = (f_j, f_{j+1}, \ldots, f_m)$ and

$$\rho(F(m,n), F(p,q)) = \exp\left[\overline{h} \sum_{j=m}^{m+n} \sum_{k=p}^{p+q} G(f_j, f_{m+k})\right].$$

Then

$$\rho(F(1,m), F(m,n))\rho(F(1,m+n), F(m+n,p))$$
$$= \rho(F(1,m), F(m,n+p))\rho(F(m,n), F(n,p))$$

and $\rho(0, F(j,k)) = \rho(F(r,s), 0) = 1$ for arbitrary $j, k, m, n, p, r, s \in \mathbb{N}$. That is, $\rho : X^{(m)} \times X^{(n)} \to \mathbb{C}, (m,n) \in \mathbb{N}^2$, is a multiplier for the group $\mathcal{G}_W(\mathbb{A}, X)$. In other words, $\mathcal{G}_W(\mathbb{A}, X, G)$ is a projective representation of $\mathcal{G}_W(\mathbb{A}, X) = \mathcal{G}_W(\mathbb{A}, X, G)|_{\overline{h}=0}$.

Corollary 3. *If $\mathcal{A}(X, W)$ and $\mathcal{A}(X, W, \hbar G)$ are the algebras generated by the $W(f)$ and $w(f) = \partial_\alpha W(\alpha f)|_{\alpha=0}$ (see definition 2), $f \in X$, with respect to the multiplication in $\mathbb{A}$ and the multiplication defined in lemma 1, respectively, then $\mathcal{A}(X, W, \hbar G)|_{\hbar=0} = \mathcal{A}(X, W)$.*

Definition 3. *The algebra $\mathcal{A}(X, W, \hbar G)$ defined in corollary 3 will be called a projective deformation of $\mathcal{A}(X, W)$.*

Remark 3. Note that the d-contraction corresponding to a projective deformation satisfies $K_d(w(f), w(g)) = G(f, g)$.

We shall now consider two examples with which we shall deal exclusively in the remainder of this paper: the (projective) deformation of a commutative algebra $\mathbb{A}_0$ and a Grassmann algebra $\mathbb{A}_1$. The first example, in which the bilinear functional G is assumed to be skew-symmetric, shall be called a *Bose-type* and the second example, in which G will be symmetric, a *Fermi-type deformation* since these examples provide a mathematical framework for (weak) Bose and Fermi quantizations, respectively. Combining $\mathbb{A}_0$ and $\mathbb{A}_1$ will finally yield a *Bose–Fermi-type deformation* that can be related to a (weak) supersymmetric quantization.

4. Bose-type deformation

Definition 4. *The deformation (cf lemma 1) $\mathcal{A}_0(X, W, \hbar G)$ of the commutative algebra $\mathcal{A}_0(X, W)$ will be called of Bose type if G is a non-degenerate skew-symmetric bilinear form.*

Remark 4. For a Bose-type deformation we shall in the following write $\mathrm{i}\Gamma/2$ instead of G. Let $W : X \to S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} : f \to W(f)$ be a continuous character. Then by a known theorem (cf [14], p 370) there exists a linear functional $w : X \to \mathbb{R}$ such that $W(f) = \exp[\mathrm{i}w(f)]$, $f \in X$. Clearly, $\mathcal{A}_0(X, W, \mathrm{i}\hbar\Gamma/2)$ is a non-commutative algebra over $\mathbb{C}$ whose product is defined by $W(f) \circ W(g) = \exp[\mathrm{i}\hbar\Gamma(f, g)/2]W(f + g)$.

The meaning of 'Bose type' will become clear by the following theorem which exhibits an intrinsic symmetrization property.

Theorem 1. *Let $\mathcal{A}_0(X, W, \mathrm{i}\hbar\Gamma/2)$ be a Bose-type deformation, and let the $f_1, \ldots, f_n$ be arbitrary elements in X. Then*

$$(1/n!) \sum_\sigma w(f_{\sigma(1)}) \circ \cdots \circ w(f_{\sigma(n)}) = w(f_1) \cdots w(f_n)$$

where the sum runs over all permutations $\sigma : (1, \ldots, n) \to (\sigma(1), \ldots, \sigma(n))$.

Proof. Let $a_j \equiv w(f_j)$, $S_+(a_1 \circ \cdots \circ a_n) \equiv (1/n!) \sum_\sigma a_{\sigma(1)} \circ \cdots \circ a_{\sigma(n)}$ and $s_j \equiv S_+(a_{j+1} \circ \cdots \circ a_{j+n}) \pmod{n+1}$. Summations will run in this proof from 1 to $n+1 \pmod{n+1}$. It is not difficult to prove that $(n+1)S_+(a_1 \circ \cdots \circ a_{n+1}) = \sum_j a_j \circ s_j = \sum_j s_j \circ a_j$. That is, $S_+(a_1 \circ \cdots \circ a_{n+1}) = \sum_j (a_j \circ s_j + s_j \circ a_j)/2(n+1)$. Further, $S_+(a_1, a_2) = a_1 \circ a_2 + a_2 \circ a_1 = 2a_1a_2 - \mathrm{i}\hbar(\Gamma(f_1, f_2) + \Gamma(f_2, f_1))/2 = 2a_1a_2$. Proceeding by induction assume that $S_+(a_{j+1} \circ \cdots \circ a_{j+n}) = a_{j+1}a_{j+2} \cdots a_{j+n} \pmod{n+1}$. Then

$$\begin{aligned} S_+(a_1 \circ \cdots \circ a_{n+1}) &= \sum\nolimits_j (a_j \circ s_j + s_j \circ a_j)/(2n+2) \\ &= (-\mathrm{i})^n \partial_{\alpha_1} \ldots \partial_{\alpha_{n+1}} \sum\nolimits_j [(W(\alpha_j f_j)) \circ (W(\alpha_{j+1} f_{j+1} + \cdots + \alpha_{j+n} f_{j+n})) \\ &\quad + (W(\alpha_{j+1} f_{j+1} + \cdots + \alpha_{j+n} f_{j+n})) \circ (W(\alpha_j f_j))]/(2n+2)\ |_{\alpha_1 = \cdots = \alpha_{n+1} = 0} \\ &= \sum\nolimits_j [2(a_j a_{j+1} \cdots a_{j+n}) - \mathrm{i}\hbar(\Gamma(f_j, f_{j+1} + \cdots + f_{j+n}) \\ &\quad + \Gamma(f_{j+1} + \cdots + f_{j+n}, f_j))/2]/(2n+2) \\ &= \sum\nolimits_j (a_j a_{j+1} \cdots a_{j+n})/(n+1) = a_1 a_2 \cdots a_{n+1}. \end{aligned}$$

□

Example 1. Let $X = \mathbb{R}^{2n}$ be the space of $2n$-tuples $z = (x, y)$, and let $X' = X$ be the space of $2n$-tuples $w = (p, q)$ of momentum and space variables $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$, respectively. That is, $w(z) = p_1 q_1 + \cdots + p_n q_n$. Let Γ be the symplectic form $\Gamma(z, \zeta) = x\eta - y\xi = x_1\eta_1 - y_1\xi_1 + \cdots + x_n\eta_n - y_n\xi_n$. Then $\mathcal{A}_0(X, W, \mathrm{i}\hbar\Gamma/2)$ corresponds to a canonical quantization of a (classical) dynamical system with n degrees of freedom.

Example 2. Let $(Y, L^2(\mathbb{R}^m), Y')$ be a Gel'fand triple where Y is a real locally convex space, let $w = (\pi, \varphi) \in X' = Y' \times Y'$, $f = (f_1, f_2) \in X = Y \times Y$ and $w(f) = \langle \pi, f_1 \rangle + \langle \varphi, f_2 \rangle$. Finally, let (here and in the following $\langle \cdot\,, \cdot \rangle$ shall denote an inner product for a function space) $\Gamma : X \times X \to \mathbb{R} : (f, g) \to \Gamma(f, g) = \langle f_1, g_2 \rangle - \langle f_2, g_1 \rangle$. Then $\mathcal{A}_0(X, W, \mathrm{i}\hbar\Gamma/2)$ corresponds to a weak canonical quantization of a real scalar field $\phi(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^m$, where $\phi(x, t) \equiv \varphi(x)$ and $\partial_t \phi(x, t) \equiv \pi(x)$ with t arbitrary fixed.

Example 3. Let $(X, L^2(\mathbb{R}), X')$ be a Gel'fand triple where X is a real locally convex space of functions $f : \mathbb{R} \to \mathbb{R}$, let $\Gamma(f, g) = \langle f, \partial_x g \rangle$, $w(f) = \langle v, f \rangle$, and $v(x) \equiv u(t, x)$ with $t \in \mathbb{R}$ fixed. Then $\mathcal{A}_0(X, W, \mathrm{i}\hbar\Gamma/2)$ corresponds to a weak canonical quantization of a real KdV field (with $u(t, x)$ satisfying a KdV equation, say, $\partial_t u - \partial_x^3 u - 3\partial_x(u^2) = 0$).

5. Fermi-type deformation

Now let $\mathbb{A}_1$ be a Grassmann algebra over $\mathbb{C}$ with a unit element $\mathbf{1}_{\mathbb{A}_1}$ and an involution, let X be a linear topological space, and let $w : X \to \mathbb{A}_1$ be linear and continuous (with respect to some topology for $\mathbb{A}_1$). That is, $w(f)w(g) = -w(g)w(f)$ and (this describes an involution) $(w(f)w(g))^* = w(g)^* w(f)^*$ for all $f, g \in X$. Furthermore, define $W(f) := \mathbf{1}_{\mathbb{A}_1} + w(f)$. It is easily verified that W is an $\mathbb{A}_1$-character. Indeed, we could write $W(f) = \exp[w(f)]$, defining the exponential of $w(f)$ by its Taylor expansion. Since $w(f)^m = 0$ for $m \geqslant 2$ this expansion boils down to the above given definition of $W(f)$. Finally, let $\mathcal{A}_1(X, W)$ denote the algebra generated by the $w(f)$, $f \in X$.

Definition 5. *If $S : X \times X \to \mathbb{C}$ is bilinear, symmetric and non-degenerate, then $\mathcal{A}_1(X, W, \hbar S/2)$ is called a (projective) Fermi-type deformation of $\mathcal{A}_1(X, W)$.*

The following theorem will exhibit an intrinsic antisymmetrization property of Fermi-type deformations.

Theorem 2. *Let $\mathcal{A}_1(X, W, \hbar S/2)$ be a Fermi-type deformation, and let the $f_1, \ldots, f_n$ be arbitrary elements in X. Then*

$$(1/n!) \sum_\sigma \mathrm{sign}(\sigma) w(f_{\sigma(1)}) \circ \cdots \circ w(f_{\sigma(n)}) = w(f_1) \cdots w(f_n)$$

where the sum runs over all permutations $\sigma : (1, \ldots, n) \to (\sigma(1), \ldots, \sigma(n))$.

Proof. Let $b_j \equiv w(f_j)$, $S_-(b_1 \circ \cdots\ b_n) \equiv (1/n!) \sum_\sigma \mathrm{sign}(\sigma)$, $(b_{\sigma(1)} \circ \cdots \circ b_{\sigma(n)})$ and $B_j \equiv \mathrm{sign}(j, j+1, \ldots, j+n) S_-(b_{j+1} \circ b_{j+2} \circ \cdots \circ b_{j+n}) \pmod{n+1}$. Summations will run in this proof from 1 to $n+1$ $(\mathrm{mod}\, n+1)$. It is not difficult to prove that $(n+1) S_-(b_1 \circ \cdots \circ b_{n+1}) = \sum_j b_j \circ B_j = (-1)^n \sum_j B_j \circ b_j$, that is,

$$S_-(b_1 \circ \cdots \circ b_{n+1}) = \sum_j (b_j \circ B_j + (-1)^n B_j \circ b_j)/(2n+2). \qquad (*)$$

Since by assumption $S(f, g) = S(g, f)\ \forall f, g \in X$, one has $w(f) \circ w(g) - w(g) \circ w(f) = 2w(f)w(g)$. Proceeding by induction assume that $S_-(b_{j+1} \circ b_{j+2} \circ \cdots \circ b_{j+n}) = b_{j+1}b_{j+2} \cdots\ b_{j+n}$ (mod $n+1$). Then by $(*)$

$$S_-(b_1 \circ \cdots \circ b_{n+1}) = \sum_j \mathrm{sign}(j, j+1, \ldots, j+n)[b_j \circ ((b_{j+1} \cdots b_{j+n}) \\ +(-1)^n (b_{j+1} \cdots b_{j+n}) \circ b_j]/(2n+2).$$

Hence

$$(b_{j+1} \cdots b_{j+n}) \circ b_j = b_{j+1} \cdots b_{j+n} b_j + \sum_{k=1}^{n} (-1)^{n-k} S(f_{j+k}, f,_j)(b_{j+1} \cdots b_{j+n})/2$$

$$= (-1)^n \left[(b_j b_{j+1} \cdots b_{j+n}) - \sum_{k=1}^{n} (-1)^{k-1} S(f_{j+k}, f,_j) \prod_{\substack{r=1 \\ r \neq k}}^{n} b_{j+r}/2 \right]$$

$$b_j \circ (b_{j+1} \cdots b_{j+n}) = (b_j b_{j+1} \cdots b_{j+n}) + \sum_{k=1}^{n} (-1)^{k-1} S(f_{j+k}, f,_j) \prod_{\substack{r=1 \\ r \neq k}}^{n} b_{j+r}/2$$

and therefore

$$S_-(b_1 \circ b_{j+2} \circ \cdots \circ b_{n+1}) = \sum_j \mathrm{sign}(j, j+1, \ldots, j+n)(b_j, b_{j+1} \cdots b_{j+n})/(n+1) \\ = b_1 b_2 \cdots b_{n+1}.$$

□

Example 4. Let $\mathcal{B}$ be a Grassmann algebra over $\mathbb{C}$ which is generated by a unit element $\mathbf{1}_\mathcal{B}$ and a set of elements b_j, $j \in \mathbb{N}$, together with their adjoints b_j^*. That is, we assume $\mathcal{B}$ to have an involution $\lambda AB \to \bar{\lambda} B^* A^*$, $\lambda \in \mathbb{C}$. Let $\mathbb{B}$ and $\mathbb{B}^*$ be the subalgebras generated by the sets $\{\mathbf{1}_\mathcal{B}, b_1, b_2, \ldots\}$ and $\{\mathbf{1}_\mathcal{B}, b_1^*, b_2^*, \ldots\}$, respectively. Now let K_d be a d-contraction of $\mathcal{B}$ such that $K_d(\mathbb{B}, \mathbb{B}) = K_d(\mathbb{B}^*, \mathbb{B}^*) = \{0\}$. That is, if $\mathbb{B}^{(m)}$, $m \in \mathbb{N}$, is the complex linear hull of monomials in the b_j with m factors, if $\mathbb{B}^{*(m)}$ is its adjoint, and $\mathbb{B}^{(0)} = \mathbb{B}^{*(0)} = \mathbf{1}_\mathcal{B}$, then

$$K_d : \mathbb{B}^{(m)} \times \mathbb{B}^{*(n)} \to \bigoplus_{j=1}^{[(m+n)/2]} \mathbb{B}^{(m-j)} \mathbb{B}^{*(n-j)} \qquad \text{for} \quad mn \neq 0 \tag{6}$$

$$K_d : \mathbb{B}^{(m)} \times \mathbb{B}^{*(n)} \to \{0\} \qquad \text{for} \quad mn = 0. \tag{7}$$

By hypothesis $AB = -BA = (B^* A^*)^*$ for $A = \alpha b_j + \beta b_k^*$, $B = \lambda b_r + \mu b_s^*$ where $j, k, r, s \in \mathbb{N}$ and $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ are arbitrary. Thus if $B_1 \in \mathbb{B}^{(m)}$ and $B_2 \in \mathbb{B}^{(n)}$ then

$$B_1 \circ B_2 = B_1 B_2 = (-1)^{mn} B_2 B_1 \tag{8}$$

$$B_1^* \circ B_2^* = B_1^* B_2^* = (-1)^{mn} B_2^* B_1^* \tag{9}$$

$$B_1 B_2^* = (-1)^{mn} B_2^* B_1. \tag{10}$$

Hence, writing $[B_1, B_2]_+ \equiv B_1 \circ B_2 + B_2 \circ B_1$,

$$[B_1, B_2]_+ = (1 + (-1)^{mn}) B_1 B_2 \tag{11}$$

$$[B_1^*, B_2^*]_+ = (1 + (-1)^{mn}) B_1^* B_2^* \tag{12}$$

$$[B_1, B_2^*]_+ = (1 + (-1)^{mn}) B_1 B_2^* + K_d(B_1, B_2^*) + K_d(B_2^*, B_1). \tag{13}$$

It follows from (8)–(10) together with (6) that if $m+n$ is even then $K_d(B_1, B_2^*)$ and $K_d(B_2^*, B_1)$ are in the commutant of $\mathcal{B}$. Further, it can easily be shown that $K_d(B_1, B_2^*) = K_d(B_2^*, B_1)$ if K_d is symmetric, that is if $K_d(b_j, b_k^*) = K_d(b_k^*, b_j)\,\forall j, k \in \mathbb{N}$. Summing up these results we have proved

Proposition 1. *Let $\mathbb{B}^{(m)}$, $m \in \mathbb{N}$, be the complex linear hull of monomials in b_j with m factors, and let $\mathbb{B}^{*(m)}$ be its adjoint. Let $B_1 \in \mathbb{B}^{(m)}$, $B_2^* \in \mathbb{B}^{*(n)}$ be arbitrary with m and n odd (but otherwise arbitrary). Then, writing $[M_1, M_2]_- \equiv M_1 \circ M_2 - M_2 \circ M_1$,*

$$[B_1, B_2]_+ = [B_1^*, B_2^*]_+ = 0 \tag{14}$$

$$[B_1, B_2^*]_+ = K_d(B_1, B_2^*) + K_d(B_2^*, B_1) \in \text{commutant}(\mathcal{B}) \tag{15}$$

$$[B_1, B_2]_- = 2B_1B_2 \tag{16}$$

$$[B_1^*, B_2^*]_- = 2B_1^*B_2^* \tag{17}$$

$$[B_1, B_2^*]_- = 2B_1B_2^* + K_d(B_1, B_2^*) - K_d(B_2^*, B_1). \tag{18}$$

If K_d is symmetric then the last equation becomes $[B_1, B_2^*]_- = 2B_1B_2^*$.

Example 5. Let $\mathcal{B}$ be the Grassmann algebra of example 4, and let

$$\psi_\alpha(t, x) = \sum_{j\in\mathbb{N}} \psi_{\alpha,j}(t, x)b_j + \sum_{j,k,l\in\mathbb{N}} \psi_{\alpha,jkl}(t, x)b_jb_kb_l + \cdots \qquad t \in \mathbb{R} \quad x \in \mathbb{R}^m$$

where $\alpha \in \{0, 1, \ldots, N\}$ and the $\psi_{\alpha,j}(t, x), \psi_{\alpha,jkl}(t, x), \ldots$ are complex-valued functions or distributions. Furthermore, let $f \in X = \mathcal{S}(\mathbb{R}^m)$ and $\psi_\alpha(f) \equiv \langle f, \psi_\alpha(t, \cdot)\rangle$ for t arbitrary fixed. Let K_d be a symmetric d-contraction for $\mathcal{B}$. That is,

$$K_d(\psi_\alpha(f), \psi_\beta(g)) = K_d(\psi_\alpha^*(f), \psi_\beta^*(g)) = 0$$

and

$$\hbar S_{\alpha\beta}(f, g)/2 \equiv K_d(\psi_\alpha(f), \psi_\beta^*(g)) = K_d(\psi_\beta^*(g), \psi_\alpha(f)) \in \text{commutant}(\mathcal{B}).$$

Let $\partial_0 \equiv \partial/\partial t$ and $\partial_r \equiv \partial/\partial x_r$, $1 \leqslant r \leqslant m$. An easy calculation using the above established relations proves that

$$[(\partial_\lambda\psi_\alpha)(f)\psi_\alpha^*(g), \psi_\beta(g)]_- = \hbar S_{\alpha\beta}(f, g)(\partial_\lambda\psi_\alpha)(f). \tag{19}$$

Let

$$Q = \sum_{\alpha=0}^{N} \int_{\mathbb{R}^m} \psi_\alpha(t, x)\psi_\alpha^*(t, x)\,\mathrm{d}x \equiv -\sum_{\alpha=0}^{N} \int_{\mathbb{R}^m} \psi_\alpha^*(t, x)\psi_\alpha(t, x)\,\mathrm{d}x$$

and assume that

$$\partial_0 Q = 0. \tag{20}$$

Further, let for all $\lambda \in \{0, 1, \ldots, m\}$

$$P_\lambda = -\mathrm{i}\sum_{\alpha=0}^{N} \int_{\mathbb{R}^m} (\partial_\lambda\psi_\alpha)(t, x)\psi_\alpha^*(t, x)\,\mathrm{d}x \equiv \mathrm{i}\sum_{\alpha=0}^{N} \int_{\mathbb{R}^m} \psi_\alpha^*(t, x)(\partial_\lambda\psi_\alpha)(t, x)\,\mathrm{d}x.$$

By partial integration it follows with (20) that $P_\lambda = P_\lambda^*$. Let $h_y(x) \equiv h(x + y)$, so that $\psi_\alpha(t, x) \equiv \psi_\alpha(\delta_x)$. If

$$S_{\alpha\beta}(\delta_y, f_x) = \delta_{\alpha\beta}\langle\delta_y, f_x\rangle\mathbf{1}_{\mathcal{B}} = \delta_{\alpha\beta} f(x - y)\mathbf{1}_{\mathcal{B}}$$

then

$$\mathrm{i}[P_\lambda, \psi_\beta(f_x)] = \overline{h}(\partial_\lambda \psi_\beta)(f_x) \qquad \mathrm{i}[P_\lambda, \psi^*_\beta(f_x)] = \overline{h}(\partial_\lambda \psi^*_\beta)(f_x). \tag{21}$$

Thus P_λ can be interpreted as the components of an energy–momentum vector associated with some Lagrangian, and Q represents a 'charge' which by (20) is a constant of motion. In particular, let $N = 3$ and assume the ψ_α to transform like the components of a Dirac spinor. Let γ_α, $0 \leqslant \alpha \leqslant 3$, be Dirac matrices. Let $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_1(x)$ be a Lagrangian where

$$\mathcal{L}_0(x) = \tfrac{1}{2}\mathrm{i} \sum_{\lambda,\alpha,\beta} (\gamma_0\gamma_\lambda)_{\alpha\beta}[\psi^*_\alpha(t,x)(\partial_\lambda\psi_\beta)(t,x) - (\partial_\lambda\psi_\alpha)(t,x)\psi^*_\beta(t,x)]$$

and where

$$\mathcal{L}_1(x) = \mathcal{L}_1(x)^* = M(\psi(t,x), \psi^*(t,x)) \qquad \psi = (\psi_0, \psi_1, \psi_2, \psi_3)$$

is a Lorentz invariant which does not contain derivatives of ψ_α and ψ^*_α and also is compatible with (20) (that is, is gauge invariant). As a consequence ψ (which assumes values in $\mathcal{B}^4$) must satisfy a (possible semilinear) Dirac equation.

6. Deformation of graded Grassmann algebras

We shall finally consider the deformation of a graded Grassmann algebra. Let X_0 and X_1 be (real) linear topological spaces, let $X = X_0 \dotplus X_1$, and let E_0 and E_1 be the projectors onto X_0 and X_1, respectively. Let w be a linear map from X into an algebra $\mathcal{A}(X, w)$ over $\mathbb{C}$ which is generated by a unit element and the set of $w(f) = w_0(f) + w_1(f)$, $w_0(f) = w(E_0 f)$, $w_1(f) = w(E_1 f)$, $f \in X$, as follows:

$$w_0(f)w_0(g) = w_0(g)w_0(f) \tag{22}$$

$$w_1(f)w_1(g) = -w_1(g)w_1(f) \tag{23}$$

$$w_0(f)w_1(g) = w_1(g)w_0(f). \tag{24}$$

Theorem 3. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the subalgebras of $\mathcal{A}(X, w)$ generated by the $w_0(f)$ and $w_1(f)$, $f \in X$, respectively. Let K_d be d-contraction for $\mathcal{A}(X, w)$ which separately also holds on $\mathcal{A}_0$ and $\mathcal{A}_1$. Further, let $\mathbb{A}_0^{(m)}$ and $\mathbb{A}_1^{(m)}$, $m \in \mathbb{N}$, be the subalgebras of $\mathcal{A}_0$ and $\mathcal{A}_1$ of monomials of degree m in the $w_0(f)$ and $w_1(f)$, $f \in X$, respectively. Then $K_d(A, B) = K_d(B, A)$ for arbitrary $A \in \mathbb{A}_0^{(m)}$ and $B \in \mathbb{A}_1^{(n)}$ where $m, n \in \mathbb{N}$ are arbitrary.*

Proof. From $AB = BA$ and $B^2 = 0$ there follows after some calculation

$$\begin{aligned} K_d(B^2, A) - K_d(A, B^2) = 0 + 0 = {} & [K_d(B, A) - K_d(A, B)]B + B[K_d(B, A) - K_d(A, B)] \\ & + AK_d(B, B) - K_d(B, B)A + K_d(B, K_d(B, A) - K_d(A, B)) + K_d(K_d(B, A) \\ & - K_d(A, B), B) + K_d(A, K_d(B, B)) - K_d(K_d(B, B), A). \end{aligned}$$

Let $a = w_0(f)$ and $b = w_1(g)$ (for arbitrary $f, g \in X$). Then, since $K_d(a, b) \in \mathbb{C}$ and $K_d(\alpha, F) = 0$ for $\alpha \in \mathbb{C}$ and $F \in \mathcal{A}(X, w)$ arbitrary, it follows that $K_d(a, b) = K_d(b, a)$. We proceed by induction. That is, we assume that $K_d(A, B) = K_d(B, A)$ for $B = b_1 \cdots b_n$, $b_j = w_1(g_j)$. Then again by some calculation, using the assumed commutativity of the elements of $\mathcal{A}_0$ and $\mathcal{A}_1$ and the properties of K_d, one obtains

$$\begin{aligned} K_d(A, bB) - K_d(bB, A) = {} & K_d(b, B)A - AK_d(b, B) + b[K_d(A, B) - K_d(B, A)] \\ & + K_d(b, K_d(A, B) - K_d(B, A)) + K_d(K_d(b, B), A) - K_d(A, K_d(b, B)) \\ & + [K_d(A, b) - K_d(b, A)]B + K_d(K_d(A, B) - K_d(B, A), B). \end{aligned}$$

By induction $K_d(A, b) = K_d(b, A)$, $K_d(A, B) = K_d(B, A)$ and, since $K_d(b, B) \in \mathbb{A}_1^{(n)}$, also $K_d(K_d(b, B), A) = K_d(A, K_d(b, B))$. So it finally follows that $K_d(A, bB) = K_d(bB, A)$. This proves the theorem. □

Corollary 4. *Let $F_0 \in \mathcal{A}_0$ and $F_1 \in \mathcal{A}_1$ be arbitrary. Then under the assumptions of theorem 3 there holds $F_0 \circ F_1 = F_1 \circ F_0$.*

Definition 6. *A deformation $\mathcal{A}(X, w, K_d)$ of $\mathcal{A}(X, w)$ will be called a Bose–Fermi-type or supersymmetric deformation if $K_d(w_0(f), w_0(g)) = -K_d(w_0(g), w_0(f))$ and $K_d(w_1(f), w_1(g)) = K_d(w_1(g), w_1(f))$.*

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Appendix. Regular Hilbert space representations of Bose-type quantizations

Let $\mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2)$ be a projective Bose-type deformation as defined in the previous section. Assume X to be real, countably normed, nuclear, and belong to a Gel'fand triple $X \subset \mathcal{H} = L^2_{real}(\mathbb{R}^m) \subset X'$ (for example, let X be the space of real rapidly decreasing functions $f : \mathbb{R}^m \to \mathbb{R}$). Then let

$$R(W(f))W(g) := W(f) \circ W(g) = \exp[\mathrm{i}\hbar\Gamma(f, g)]W(f + g). \tag{A1}$$

Since the skew-symmetric bilinear functional $\Gamma : X \times X \to \mathbb{R}$ was assumed to be continuous in each argument there exists by the kernel theorem (cf [15]) a linear map $K : X \to X$ so that $\Gamma(f, g) = \langle Kf, g\rangle$ for all $f, g \in X$ where $\langle\cdot\,,\cdot\rangle$ denotes the inner product for $\mathcal{H}$. Let ν_Φ be a bounded (complex) measure on X, and let $\Phi(w) = \int_X W(g)\,\mathrm{d}\nu_\Phi(g)$. Then by (A3)

$$R(W(f))\Phi(w) = \int_X \exp(\mathrm{i}\langle w + \hbar Kf/2, g\rangle)W(g)\,\mathrm{d}\nu_\Phi(g) = \Phi(w + \hbar Kf/2)$$

where $w(g) \equiv \langle w, g\rangle$, etc. In view of the finite-dimensional case, that is, for $X = \mathbb{R}^{2n}$, one might be tempted to assume that the map $R(W(f)) : \Phi(w) \to R(W(f))\Phi(w) = \Phi(w + \hbar Kf/2))$ is a unitary map in some Hilbert space $L^2(X', \mu)$ (containing functions $\Phi(w)$). However, this would be true only if the range of K were an invariant subspace relative to the measure space $(X', \mathcal{B}(X'), \mu)$, $\mathcal{B}(X')$ being the family of Borel sets of X'. That is, one should have $\mathrm{d}\mu(w) = \mathrm{d}\mu(w + f)$ for all $w \in X'$ and $f \in X$. Since, in general, X' will not bear a locally compact topology, such a condition is too stringent: it could happen (cf [16]) that the only measure to comply with it is the zero measure. The following proposition will show how to obtain a representation in which the $W(f)$, $f \in X$, are unitary.

Proposition 2. *Let $X \subset \mathcal{H} \subset X'$ be a Gel'fand triple, and let $\mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2)$ be a Bose deformation. Then there exists a (non-unique) positive, σ-additive, normed and quasi-invariant measure μ on X', and a function $\chi_\mu : X' \times X' \to \mathbb{R}_+$ with the property*

$$\chi_\mu(w, w + f)\chi_\mu(w + f, w + f + g) = \chi_\mu(w, w + f + g) \tag{A2}$$

for all $w \in X'$ and $f, g \in X$, so that the linear map $R_\mu : \mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2) \to \mathcal{B}(L^2(X', \mu))$:

$$R_\mu(W(f))\Phi(w) = \chi_\mu(w, w + \hbar Kf/2)^{1/2}W(f)\Phi(w + \hbar Kf/2) \tag{A3}$$

is a *-homomorphism, and $\{R_\mu(W(f))|f \in X\}$ is a group of unitary operators in $\mathcal{B}(L^2(X', \mu))$.

Proof. The assumed properties of X guarantee (cf [16]) the existence of measures μ on X' which are as described. The quasi-invariance implies the existence of a Radon–Nikodym derivative $\chi_\mu(w, w+f) = \mathrm{d}\mu(w+f)/\mathrm{d}\mu(w) \geqslant 0$ which satisfies (28). Let $\Phi, \Psi \in L^2(X', \mu)$. Then by definition

$$\begin{aligned}
\langle \Phi, R_\mu(W(f))\Psi\rangle &= \int_{X'} \overline{\Psi(w)} W(f)\Phi(w + \hbar K f/2)\chi_m u(w, w + \hbar K f/2)^{1/2}\,\mathrm{d}\mu(w) \\
&= \int_{X'} \overline{W(-f)\Psi(w - \hbar K f/2)\chi_m u(w, w + \hbar K f/2)^{1/2}}\,\Phi(w)\,\mathrm{d}\mu(w) \\
&= \langle R_\mu(W(-f)\Psi, \Phi\rangle = \langle R_\mu(W(f)^*)\psi, \phi\rangle. \qquad \text{(A4)}
\end{aligned}$$

Hence for all $f \in X$ (using the same notation for taking the adjoint in $\mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2)$ and $\mathcal{B}(L^2(X', \mu))$)

$$R_\mu(W(f))^* = R_\mu(W(-f)) = R_\mu(W(f)^*).$$

Substituting in (30) $\Psi = R_\mu(W(f))\Phi$ one obtains ($\|\cdot\|$ denotes the norm in $L^2(X', \mu)$) $\|R_\mu(W(f))\Psi\| = \|\Psi\|$. Further, by (28)

$$\begin{aligned}
R_\mu(W(f))(R_\mu(W(g))) &= \chi_\mu(w, w + \hbar K f/2)^{1/2}\chi_\mu(w + \hbar K f/2, w + \hbar K(f+g)/2)^{1/2} \\
&\quad \times \exp(\mathrm{i}\hbar\langle Kf, g\rangle/2)W(f+g)\Phi(w + \hbar K(f+g)/2) \\
&= \chi_\mu(w, w + \hbar K(f+g)/2)^{1/2} \\
&\quad \times \exp(\mathrm{i}\hbar\langle Kf, g\rangle/2)W(f+g)\Phi(w + \hbar K(f+g)/2) \\
&= R_\mu(W(f) \circ W(g))\Phi(w).
\end{aligned}$$

Since $W(f)^{-1} = W(-f) = W(f)^*$, $W(0) = \mathbf{1}_{\mathcal{A}}$ and $\chi_\mu(w, w) = 1$, there follows

$$R_\mu(W(f) \circ W(f)^{-1})\Phi(w) = R_\mu(\mathbf{1}_{\mathcal{A}})\Phi(w) = \Phi(w)$$

so that finally

$$R_\mu(W(f)^*) = R_\mu(W(f)^{-1}) = R_\mu(W(f))^{-1} = R_\mu(W(f))^*.$$

This proves the assertion. □

Remark 5. The non-uniqueness of μ reflects (cf [16]) the existence of (infinitely many) different non-equivalent representations R_μ. These representations also make sense for finite systems, for example, when μ is chosen a Gauss measure on the (finite-dimensional) phase space. R_μ is called a *left-regular μ-representation* of $\mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2)$ (by replacing $\hbar$ by $-\hbar$ one obtains a *right-regular μ-representation* of $\mathcal{A}(X, W, \mathrm{i}\hbar\Gamma/2)$).

Remark 6. To represent by R_μ larger classes of functions, one can use Fourier transforms (recall that the $W(f)$ are characters). For example, if

$$F(w(f_1), \ldots, w(f_n)) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{F}(\alpha_1, \ldots, \alpha_n)W(\alpha_1 f_1 + \cdots + \alpha_n f_n)d(\alpha_1, \ldots, \alpha_n)$$

then

$$\begin{aligned}
R_\mu(F(w(f_1), \ldots, w(f_n)))\Phi(w) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{F}(\alpha_1, \ldots, \alpha_n)W(\alpha_1 f_1 + \cdots + \alpha_n f_n) \\
&\quad \times \Phi_\chi(w, w + \hbar K(\alpha_1 f_1 + \cdots + \alpha_n f_n)/2)d(\alpha_1, \ldots, \alpha_n)
\end{aligned}$$

where $\Phi_\chi(w, w') = \chi_\mu(w, w')^{1/2}\Phi(w')$. In particular,

$$R_\mu(w(f_1), \ldots, w(f_n))\Phi(w) = \prod_{j=1}^{n}(w(f_j) - \mathrm{i}\overline{h}\langle Kf_j, \delta/\delta g\rangle/2)\Phi_\chi(w, g)|_{g=0}$$

where $\langle k, \delta/\delta g\rangle\lambda(g) = (\delta\lambda(g)/\delta g)(k) = \lim_{\varepsilon\downarrow 0}[\varepsilon^{-1}(\lambda(g+\varepsilon k) - \lambda(g))]$. If for example ν_Φ is a positive bounded measure on X and $\Phi(w) = \int_X W(g)\,\mathrm{d}\nu_\Phi(g)$ then

$$R_\mu(w(f_1), \ldots, w(f_n))\Phi(w) = \chi_\mu(w, w + \overline{h}K(\alpha_1 f_1 + \cdots + \alpha_n f_n)/2)^{1/2} \times \int_X \prod_{j=1}^{n}(w(f_j) + \overline{h}\langle Kf_j, g\rangle/2)W(g)\,\mathrm{d}\nu_\Phi(g).$$

A sufficient condition for the existence of the integral is that for arbitrary $N \in \mathbb{N}$ the function $g \to \|g\|^N$ is in $L^1(X, \nu_\Phi)$.

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